Enemies or Allies: Pricing counterparty credit risk for synthetic CDO tranches

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Enemies or Allies: Pricing Counterparty Credit Risk for Synthetic CDO Tranches

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Abstract

This research aims to construct a model for pricing counterparty credit risk (CCR) for synthetic collateralized debt obligation (CDO) tranches by considering the relationship between the counterparty and the credit portfolio. A stochastic intensity model is adopted to describe the default event of the counterparty, and a two-factor Gaussian copula model is applied to account for the relationship between the counterparty and underlying credit portfolio. By analyzing the data of CDX NA IG index tranches, we find that the relationship has a significant influence on the credit value adjustment (CVA) for index tranches and, hence, that it should not be ignored when a contract is initiated. In addition, we discover that the influence has opposite effects and asymmetrical magnitude with respect to the protection buyers and protection sellers.

Keywords: counterparty credit risk; synthetic CDO tranches; CDX NA IG index tranches; Gaussian copula model; credit value adjustment

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1 Introduction

Not until the financial tsunami burst in July 2007 did multi-name credit derivatives, such as collateralized debt obligations (CDOs), lose their popularity. Since then, CDOs have become one of the targets of public criticism for the subprime mortgage crisis. Watson (2008) gave a detailed review on the origins of this crisis and mentioned the dark side of such securitized products. Both market practitioners and regulators are seeking to find safer ways to trade such over-the-counter products. Predicting default probabilities by using refined credit-risk models is a commonly discussed potential solution (see, for example, Chen, Liao, and Lu, 2011).

Among several solutions that have been suggested, managing counterparty credit risk (CCR) may be the most crucial one. The CCR of a derivative product refers to the risk that the counterparty will fail to fulfill the obligation specified by his contract. In the case of synthetic CDOs, the protection buyer, who commits periodic payments and an upfront fee, and the protection seller, who offers protections for a portfolio of credit default swaps (CDSs) that are within a certain tranche, are counterparties with respect to each other.

An old Chinese saying goes, "Even brothers keep careful accounts." Handling CCR is definitely the most sophisticated way to keep CDOs actively trading in the market without throwing out the baby with the bathwater. Much research has been devoted to pricing derivatives with CCR. By using the barrier model, Hull and White (2000) evaluated CDS contracts with counterparty risk. Jarrow and Yu (2001) built a correlated intensity model to describe CCR and investigated its impact on defaultable security pricing. With a firm-value-based default barrier model, Brigo and Tarenghi (2004) priced CDS and equity swap under CCR. Applying the approach mentioned by Collin-Dufresne et al. (2002), Leung and Kwok (2005) priced CDS contracts with counterparty risk.

Another approach called "credit value adjustment" (CVA) has drawn lots of academic attention in recent years. According to Zhu and Pykhtin (2007), CVA, which is the portfolio value difference between the risk-free evaluation and the CCR-considered evaluation, can be used to represent the market value of CCR. In addition to CCR in CDS, Brigo and Masetti (2006) discussed how to deal with CCR in interest-rate swaps and equity-return swaps. Studies that use CVA to handle CCR in other financial assets can be found in Brigo, Pallavicini, and Papatheodorou (2009) on interest-rate swaps and Brigo and Bakkar (2009) on energy and commodity products.

However, little literature is available on the pricing of CCR in synthetic CDO tranches. This may be because any correlation is more difficult to identify and even harder to cope with when it comes to cases of multi-name products. The purpose of this study is to bridge the gap in pricing CCR in over-the-counter synthetic CDOs. It should be noted that in order to simplify things, we assume the party who computes the CVA is default free (i.e., we consider a unilateral case of CCR). Note that such an analysis can be viewed as approximate to the situation where the party has much better credit quality than the counterparty. The dependence between the underlying credit portfolio and the counterparty is the key risk factor modeled in our framework.

In our framework, a stochastic intensity model is adopted to characterize the default event of the counterparty. To give a detailed analysis of the impact of this relationship, we introduce a two-factor Gaussian copula model to account for this relationship between the counterparty and the underlying credit portfolio. The additional risk factor in the copula model represents the credit quality of the counterparty. We assume a correlation between this additional risk factor and the original common factor (i.e., factor correlation) to detect dependence risk.

We analyze the case of the CDX NA IG index tranches to show the applications of our theoretical results. Practically, people would not classify an index tranche as a tranche of a synthetic CDO. While the former is a standardized insurance contract and not funded by a portfolio of CDSs, the latter is a credit-linked note and funded. Nevertheless, the cash flow of an index tranche is the same as the corresponding synthetic CDO tranche, and they can be priced by following the same rule (see, for example, Hull and White, 2004; Wang, Rachev, and Fabozzi, 2006; Hull, 2009).

In our study, we find that dependence has a relevant impact on CVA. This impact is analyzed by changing values of factor correlation across different tranches by means of some numerical examples. In addition, we find that the impact is different for protection buyers and protection sellers. For protection buyers, the correlation has a negative impact on CVA, and therefore, they have no need to worry about wrong-way risk. For protection sellers, on the other hand, the correlation has a positive impact on CVA. Because they are exposed to wrong-way risk when selling protection on equity tranches, they have to handle counterparty risk more carefully. Therefore, we wish to emphasize that the relationship between the underlying credit portfolio and the counterparty has a significant effect on CVA for synthetic CDO tranches and should not be ignored.

The organization of this paper is as follows: In section 2, the general valuation formula of unilateral CCR is reviewed. In section 3, a theoretical framework that includes the intensity model for default of the counterparty and the copula model for the underlying credit portfolio is described. Section 4 features an application of the theoretical model to solve CVA in synthetic CDO. In sections 5 and 6, model calibration and numerical examples are illustrated, and in the final section, we present our conclusions.

2 Arbitrage-Free Pricing of Counterparty Credit Risk

In this section, we briefly review the general pricing theory of CCR. It is worth mentioning at the onset that whether or not the party valuing CCR takes his own credit risk into consideration will lead to different values of CCR. If the party regards himself as default free, the CCR valuation is said to be unilateral; otherwise, it is said to be bilateral. Throughout this study, we consider the case of unilateral CCR. For a discussion of general bilateral CCR pricing, please refer to, for instance, Brigo and Capponi (2009) and Gregory (2009).

In what follows, we act as the default-free investor so that the results derived below are from the investor's viewpoint. We define τ_C as the default time of the counterparty, and we suppose that we are in a filtered probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$. The filtration $\mathcal{G}_t$ comprises all information available up to time t, and $\mathcal{F}_t \subset \mathcal{G}_t$, a complete and right-continuous filtration, denotes the default-free information set. $\mathbb{Q}$ is the risk-neutral probability measure.

Suppose we enter into a derivative contract that has a final maturity T. In the situation where $\tau_C > T$, there is no default on the contract, and the obligation is fulfilled. Where $\tau_C < T$, the counterparty fails to make payments to the investor. When the counterparty defaults prior to the final maturity, we assume the following situation. At the default time τ_C, the net present value (NPV) of the remaining cash flows that would be received if the counterparty had not defaulted is calculated. Whenever the NPV is negative with respect to the investor (and, correspondingly, positive to the counterparty), this amount is completely paid by the investor. On the other hand, where the NPV is positive with respect to the investor (and, thus, negative with respect to the counterparty respectively), the NPV is partially received as the recovery fraction REC_C.

$\Pi^D(t,T)$ denotes the theoretical discounted payoff of the financial contract, and $Cashflow(u,s)$ denotes discounted total cash flows at time u without default between u and s. $NPV(\tau_C)$ denotes the expected discounted cash flow conditional on all information available up to τ_C, so that $NPV(\tau_C) := E_{\tau_C}\left[Cashflow(\tau_C, T)\right]$ and

$$\begin{aligned}\Pi^D(t,T) = I_{(\tau_C>T)}&Cashflow(t,T)\\ &+ I_{(\tau_C<T)}[Cashflow(t,\tau_C)\\ &+ D(t,\tau_C)\left(REC_C[NPV(\tau_C)]^+ - [-NPV(\tau_C)]^+\right)],\end{aligned} \tag{1}$$

where $E_t[\,,\,]$ denotes the expectation conditioning on all information up to t under the risk-neutral probability measure, and $I_{(\,,\,)}$ is the indicator function. $D(u,s)$ is the stochastic discount function at u for maturity s. Notice that this is the general discounted payoff formula for defaultable financial contracts. The first term on the right-hand side of the equation is the total discounted cash flow received, contingent on there not being any default prior to maturity. When an early default occurs, the payments before the default are received (second term), and a recovery fraction is received in the case of positive residual NPV (third term). If, however, the residual NPV is negative, it is completely lost (the last term). Please note that the expression will reduce to risk-neutral pricing payoff when the CCR is not considered.

Brigo and Masetti (2006) derive the formula for defaultable security pricing, and we summarize their results in the following theorem.

Theorem 2.1. The time t price of a defaultable financial contract with final maturity T assuming that the counterparty has not defaulted before t is

$$\begin{aligned} E_t\left(\Pi^D(t,T)\right) &= E_t\left(\Pi(t,T)\right) \\ &\quad - (1-REC_C)E_t\left(I_{(t<\tau_C<T)}D(t,\tau_C)[NPV(\tau_C)]^+\right), \end{aligned} \tag{2}$$

where $\Pi(t,T)$ is the default-free discounted payoff when counterparty risk is not taken into account, and the recovery rate REC_C is assumed to be deterministic. The price subject to counterparty risk is the default-free price minus a call option with zero strike written on the residual NPV giving nonzero values only if $\tau_C < T$.

For a rigorous proof of this theorem, please see Brigo and Masetti (2006).

The additional amount of the default-free discounted payoff to the discounted payoff with potential counterparty risk is the default option of the counterparty. It can be viewed as the price of counterparty risk, and it is often referred to as counterparty credit risk credit value adjustment (CCR-CVA, or CVA for short). It is always nonnegative, and it represents the cost of accessing the transaction. Furthermore, we should emphasize that the residual NPV should be calculated based on information of both market quantities and default event of the counterparty. This is critical in the situation where the dependence of the payoff on counterparty credit quality is significant. Such dependence is the so-called "wrong/right-way risk".

Note that the CVA can be rewritten as:

$$\begin{aligned} CVA(t,T) &= Loss_C E_t\left(I_{(t<\tau_C<T)}D(t,\tau_C)[NPV(\tau_C)]^+\right) \\ &= Loss_C \int_t^T E_t\left(D(t,u)[NPV(u)]^+ | \tau_C = u\right) d_u\mathbb{Q}(\tau_C \in [t,u]). \end{aligned} \tag{3}$$

where $Loss_C = 1 - REC_C$. In this expression, the term in the integral is computed based on all information of market quantities up to t and the knowledge that the counterparty will default at some future time u. If we assume independence between exposure and the counterparty's credit quality, equation (3) can be further simplified as

$$CVA(t,T) = Loss_C \int_t^T E_t\left(D(t,u)[NPV(u)]^+\right) d_u\mathbb{Q}(\tau_C \in [t,u]). \tag{4}$$

In this case, the term being integrated is called expected exposure, and it is independent of the counterparty's credit quality. This independence simplifies the calculation of CVA, but such a simplification may prove dangerous where the dependence between the payoff and counterparty is too significant to be ignored. Therefore, for the purpose of this study, we have not simplified our calculation in this manner.

3 The Modeling Framework

In order to calculate the CVA for synthetic CDO tranches, we have to model not only the default probabilities of those credits in the credit portfolio underlying the contract that takes account of the counterparty's credit quality, but also the counterparty itself. We have constructed an intensity model and the two-factor Gaussian copula model for the counterparty and the underlying credit portfolio respectively. The intensity model is stochastic, and a deterministic function is added on in order to fit market implied survival probabilities. On the other hand, the two-factor Gaussian copula model, compared to a traditional one-factor case, makes use of an additional risk factor to capture the effect of the counterparty's credit quality.

3.1 The Shifted Stochastic Intensity Model

In this subsection, we assume default to be a random event that can be characterized by a Poisson process with stochastic intensity. We use M_t to denote the Poisson process with stochastic intensity γ_t, and we define the counterparty default time τ_C as the first jump time of the process so that:

$$\tau_C := \inf_t \{M_t > 0\}. \tag{5}$$

Let us now construct the intensity process. We assume γ_t is $\mathcal{F}_t$ adapted (i.e., γ_t is non-random conditional on the filtration $\mathcal{F}_t$) and denote by $\mathcal{D}_t$ the filtration generated by γ up to time t. In other words, $\mathcal{D}_t = \sigma(\{\gamma_s; s \leq t\})$. We assume that

γ is driven by the following stochastic system:

$$\begin{aligned} \gamma_t &= \beta(t; \kappa, \mu, \nu, \lambda_0) + \lambda_t, \\ d\lambda_t &= \kappa(\mu - \lambda_t)dt + \nu\sqrt{\lambda_t}dW_t, \end{aligned} \tag{6}$$

where $\beta(,)$ is a deterministic function of time and the parameter vector $(\kappa, \mu, \nu, \lambda_0)$, and it is assumed to be positive and integrable. W_t represents one-dimensional Brownian motion under the risk-neutral probability measure $\mathbb{Q}$. This model is sometimes referred to as the "Shifted Squared Root Diffusion" (SSRD) or "Shifted Cox, Ingersoll and Ross" (CIR++) model (see, for example, Brigo and Mercurio (2006)). With this model, we can derive the time t risk-neutral survival probability, that is, the probability that the counterparty will not default prior to some future time T, given the information available up to time t. The formula that conveys this can be rendered more precisely, as follows:

$$\begin{aligned} \mathbb{Q}(\tau_C > T|\mathcal{G}_t) &= I_{(\tau_C>t)}E_t\left[I_{(\tau_C>T)}\right] \\ &= I_{(\tau_C>t)}E_t\left[\mathbb{Q}(\tau_C > T|\mathcal{D}_T)\right] \\ &= I_{(\tau_C>t)}E_t\left[\exp\left(-\int_t^T \gamma_u du\right)\right] \\ &= I_{(\tau_C>t)}\exp\left(-\int_t^T \beta(u; \kappa, \mu, \nu, \lambda_t)du\right)E_t\left[\exp\left(-\int_t^T \lambda_u du\right)\right] \\ &= I_{(\tau_C>t)}\exp\left(-\int_t^T \beta(u; \kappa, \mu, \nu, \lambda_t)du\right)Survival(t,T), \end{aligned} \tag{7}$$

where $Survival(t,T)$ is the time t survival probability formula for the non-shifted CIR model. The formula can be solved analytically as follows:

$$Survival(t,T) = Survival(t,T; \kappa, \mu, \nu, \lambda_t) = A(t,T)e^{-L(t,T)\lambda_t}, \tag{8}$$

where

$$\begin{aligned} L(t,T) &= \frac{2(\exp[h(T-t)]-1)}{2h + (\kappa+h)(exp[h(T-t)]-1)}, \\ A(t,T) &= \left(\frac{2hexp[(\kappa+h)(T-t)/2]}{2h+(\kappa+h)(exp[h(T-t)]-1)}\right)^{\frac{2\kappa\mu}{\nu^2}}, \\ h &= \sqrt{\kappa^2 + 2\nu^2}. \end{aligned}$$

The above formula enables us to define the cumulative distribution function for the default time $H_C(\,,\,)$ under the risk-neutral probability measure as follows:

$H_C(T) = 1 - \mathbb{Q}(\tau_C > T|\mathcal{G}_0)$. In the two-factor copula model that we will detail in the next subsection, this is a key factor in capturing the counterparty credit quality. Note further that we only need to compute $\int_t^T \beta(u;\kappa,\mu,\nu,\lambda_t)du$. rather than $\beta(u;\kappa,\mu,\nu,\lambda_t)$ itself. This will reduce the computation dramatically.

3.2 The Two-Factor Gaussian Copula Model

Let us consider a credit portfolio consisting of M reference credits, and let us use τ_i to denote the default time of the *i-th* credit. We assume the credit pool is homogeneous, i.e., the reference credits are identical. Similar to our method of proceeding in the previous subsection, we assume $\tau_i s$ to be the first jump time of the Poisson process $M_i(t)$ with stochastic intensity h_t, and we assume h_t to follow the square-root process:

$$dh_t = \alpha(\theta - h_t)dt + \sigma\sqrt{h_t}dW_t^i, \tag{9}$$

where W_t^i is a one-dimensional Brownian Motion under the risk-neutral probability measure $\mathbb{Q}$. It is independent of W_t in equation (6), and for $i \neq j$, W_t^i is independent of W_t^j. We can solve the risk-neutral survival probability for credit i by equation (8) and define $H(\ ,\ )$ as the common risk-neutral cumulative distribution function conditional on all current available information of τ_i for all i, which implies that the random variable $U_i = H(\tau_i)$ is uniformly distributed. To construct the dependence structure of default times, we can then introduce the two-factor model:

$$X_i = a \times F + b \times C + \sqrt{1 - a^2 - b^2 - 2\rho ab} \times Z_i, \tag{10}$$

where we assume a and b to be bounded by the interval [-1,1] and (F, C) to be common among all reference credits that are bivariate normally distributed with zero mean, unit variance and correlation coefficient ρ. For $i = 1, ..., M$, Z_i are a sequence of independent and identically distributed standard normal random variables that are uncorrelated with common factors F and C. To connect marginal default time distributions, we adopt the following copula function $C(\ ,\ )$:

$$C(u_1, u_2, ..., u_M) = \Phi_M\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2), ..., \Phi^{-1}(u_M)\right). \tag{11}$$

Here, $\Phi_M(\, , \,)$ denotes the M-dimensional cumulative normal distribution function with mean zero, variance one and pairwise correlation $\rho_X = a^2 + b^2 + 2\rho ab$. $\Phi(\, , \,)$ denotes the univariate standard normal cumulative distribution function. This means that we can map the marginal distributions to the joint distribution through a multivariate normal correlation structure. The joint distribution of the default times can thus be:

$$\mathbb{Q}(\tau_1 < t_1, ..., \tau_M < t_M) = \Phi_M \left(\Phi^{-1}[H(t_1)], ..., \Phi^{-1}[H(t_M)] \right). \tag{12}$$

This dependence structure is quite intuitive; the defaults of two underlying credits are related by the two common factors that can be interpreted as the macroeconomic factor and the variable representing the counterparty's credit quality.

It is easy to check that X_i is standard normally distributed, and the correlation between different X_is is ρ_X, which is the default correlation between two underlying credits. Furthermore, the default correlation between the counterparty and underlying credits can be computed as the correlation between X_i and C, which is $a\rho + b$. It is always bounded by the interval $[-1, 1]$ to reveal the fact that one should not be perfectly correlated with some portfolio due to the diversification effect.

The distribution introduced above turns out to be sufficient for the multi-name default swaps and basket product valuation. However, when it comes to pricing derivatives depending on the default rate fluctuation, for instance, n-th to default swap or CDO tranches, we should further specify the default rate distribution at any time horizon.

In order to make this clearer, let us consider the example of a derivative product whose payoff depends on the default rate at some future time horizon T. According to the risk-neutral pricing theory, the current price of the derivative is the risk-neutral expectation of payoff function divided by the bank account numeraire at T. This can be represented as follows:

$$Price(0) = \frac{Price(0)}{B(0)} = E_0 \left[\frac{Payoff(n,T)}{B(T)} \right], \tag{13}$$

where $Payoff(n,T)$ is the payoff function, which depends on the number of defaults n, and $B(u)$ denotes the value of a bank account at time u. In order to compute the expectation, if the bank account is assumed to be deterministic, we only need to know the risk-neutral distribution of the default rate at time T. By (10), we know that conditional on F and C, the X_is are independent of each other, as are the τ_is. We can make use of this property to construct the default rate distribution. Firstly, we should compute the risk-neutral default probability of reference credit i conditional on the value of F and C:

$$\begin{aligned} q(T) &= \mathbb{Q}(\tau_i < T|F, C) \\ &= \mathbb{Q}(H(\tau_i) < H(T)|F, C) \\ &= \mathbb{Q}(\Phi(X_i) < H(T)|F, C) \\ &= \Phi\left(\frac{\Phi^{-1}[H(T)] - aF - bC}{\sqrt{1 - a^2 - b^2 - 2\rho ab}}\right). \end{aligned} \tag{14}$$

Due to conditional independence, whether or not defaults prior to time T can be represented as a sequence of i.i.d. Bernoulli random variables. This implies that the probability that n will default among M prior to time T conditional on F and C is:

$$\pi(n, M, T) = \mathbb{Q}(n, M, T|F, C) = \frac{M!}{n!(M-n)!}\left[q(T)\right]^n \left[1 - q(T)\right]^{M-n}. \tag{15}$$

Once the conditional probability of the default rate is established, we are about to complete the pricing of the derivative. Let us go back to equation (13) and rewrite it with the following iterated expectation:

$$\begin{aligned} Price(0) &= E_0\left[\frac{Payoff(n, T)}{B(T)}\right] \\ &= E_0\left(E_0\left[\frac{Payoff(n, T)}{B(T)}|F, C\right]\right). \end{aligned} \tag{16}$$

Equation (16) is valid based on the tower property. However, the price would differ if we had the information about C. For instance, if we knew that $C = c_0$, the current price of the derivative would be:

$$\begin{aligned} Price(0|c_0) &= E\left(\sum_{n=0}^{M}\frac{Payoff(n, T)}{B(T)}\pi(n, M, T)|C = c_0\right) \\ &= \int_{-\infty}^{\infty}\sum_{n=0}^{M}\frac{Payoff(n, T)}{B(T)}\pi(n, M, T)g(f|c_0)df, \end{aligned} \tag{17}$$

where $g(f|c_0)$ is the conditional density of F given $C = c_0$. The expression can be solved efficiently by means of either numerical integration or Monte Carlo simulation.

We note that, compared to the one-factor Gaussian copula model, a new common risk factor "C" representing an additional source of uncertainty has been added on. This allows for more flexibility. When discussing CCR valuation, "C" serves as a proxy for the impact of the counterparty's credit quality on the underlying portfolio. Formally, we can set this as:

$$H_C(\tau_C) = \Phi(C). \tag{18}$$

This gives us a way to account for the impact of the counterparty whenever default information is obtained. Our approach is quite innovative in that our model deals with default rate distribution conditional on the counterparty's credit quality rather than on the joint default distribution of reference credits and the counterparty, which is often considered in the CCR modeling literature. As shown in the previous section, this conditional distribution combined with the survival probability curve of the counterparty is sufficient for the evaluation of CVA. Before closing this section, we would like to point out that this model can be easily extended to either non-Gaussian copula assumptions or time-inhomogeneous parameterization.

4 Application to Synthetic CDO Tranches

A synthetic CDO tranche is similar to a CDS in that one party (the protection seller) offers protection for losses caused by a particular credit event in exchange for periodic payments and an upfront fee (if any) that is received from the other party (the protection buyer). The difference is that CDS contracts offer protection for the credit event of one entity, so that it covers total losses, while CDO tranches offer protection for only a portion of a portfolio of short position CDSs. This portion is defined by the attachment point (A) and detachment point (B). In the following, we briefly review the arbitrage-free pricing formula of CDO tranches without CCR(in the first subsection) and then move on to discuss the CVA calculation (in the second subsection).

4.1 Arbitrage-Free Valuation of Synthetic CDO Tranches

We consider a CDO tranche written on a portfolio containing M CDSs with attachment point A, detachment point B, the first reset time T_0, and payment times

$[T_1, T_2, ..., T_b]$. Such a contract is quoted in the form of an upfront fee that has to be paid at the beginning in conjunction with a periodic running spread S (usually 100 or 500 basis points). The principle of pricing swap-like contracts consists in making the contract "fair" from both parties' points of view. In the case of CDO tranches, this amounts to equating the expected cash flows received by both protection seller and buyer. To this end, we define $Default_{A,B}(t, T_0, T_b)$ as the expected discounted protection payment of the protection seller, $Premium_{A,B}(t, T_0, T_b)$ as the expected discounted periodic payment from the protection buyer when the running spread is unity, and $U_{A,B}(t, T_0, T_b)$ as the upfront fee that needs to be paid (if it is positive) or received (if it is negative) by the protection buyer. The contract is said to be fair if and only if:

$$U_{A,B}(t, T_0, T_b) + S \times Premium_{A,B}(t, T_0, T_b) = Default_{A,B}(t, T_0, T_b). \tag{19}$$

In the following discussion, we assume deterministic interest rates and a constant recovery rate R. In addition, following Brigo and Mercurio (2006), we adopt a postponed CDS payoff in order to simplify this calculation. The following theorem formulates the arbitrage-free pricing formula for CDO tranches:

Theorem 4.1. At any time t, the arbitrage-free values for the default and premium legs of a CDO tranche with attachment point A, detachment point B, first reset time T_0, payment time $[T_1, ...T_b]$, and unit notional principal are expressed by:

$$\begin{aligned}
Default_{A,B}(t, T_0, T_b) &= \sum_{w(t)}^{b} P(t, T_i) \left[E_t \left(L_{A,B}[n(T_i)] - L_{A,B}[n(T_{i-1})]\right)\right], \\
Premium_{A,B}(t, T_0, T_b) &= \sum_{w(t)}^{b} P(t, T_i)\delta_i \left[1 - \frac{E_t \left[L_{A,B}[n(T_i)] + L_{A,B}[n(T_{i-1})]\right]}{2}\right].
\end{aligned} \tag{20}$$

Here, $P(u, s)$ denotes the time u price of a zero-coupon bond maturing at s, δ_i is the year fraction of the interval $[T_{i-1}, T_i]$, and $w(t)$ returns the first i such that T_i is the first payment time following t. The tranche loss, $L_{A,B}[n(u)]$, is defined as:

$$L_{A,B}[n(u)] = \left[(B - A)I_{\left(\frac{n(u)\times(1-R)}{M} < A\right)} + \left(B - \frac{n(u) \times (1 - R)}{M}\right)I_{\left(A \leq \frac{n(u)\times(1-R)}{M} \leq B\right)}\right] / (B - A),$$

where $n(u)$ denotes the number of defaults at time u.

From the protection seller's point of view, the value of the CDO tranche is calculated by: [1]

$$S \times Premium_{A,B}(t, T_0, T_b) - Default_{A,B}(t, T_0, T_b). \tag{21}$$

It is clear that, in this expression, the only unknown quantity is the expected tranche outstanding notional principal. This quantity can be calculated based on the two-factor Gaussian copula model proposed above or through more complex dynamic models. [2]

4.2 Credit Value Adjustment for Synthetic CDO Tranches

In this subsection, we will focus on how to compute the CVA for a CDO tranche. First, we suppose that the investor has entered into a CDO contract with first reset date T_0, payment dates $[T_1, ...T_b]$, attachment point A, detachment point B, and notional principal I . We further assume that the investor is acting as the protection seller; that is, he offers protection to some counterparty C on a loss of $A\%$ to $B\%$ until the final maturity T_b or until the portfolio loss exceeds $B\%$ if party C does not default. He bears the risk that party C defaults before the final payment date T_b. A similar approach can be applied if we are from the protection buyer's point of view. To save space, we omit the case of the protection buyer in the following deductions.

If you recall the CVA formula reviewed in section 2 (expression (4)), you will see that all we need to compute is the expectation in the integral. The expectation can be evaluated by using the two-factor Gaussian copula model found in equations (10) and (11) and the arbitrage-free formula for CDO tranches (equation (20)). We are now able to derive the NPV quantity for the expectation. It is the time t value of the CDO tranche given all information available up to time t. The

[1] For the protection buyer, the value of the same CDO tranche is simply the negative of the value for the protection seller; that is:

$$Default_{A,B}(t, T_0, T_b) - S \times Premium_{A,B}(t, T_0, T_b).$$

[2] See, for example, Hull and White (2008) and Brigo, Pallavicini, and Torresetti (2011) for dynamic models of multi-name credit derivatives.

information set must contain the default information for all credits underlying the portfolio. That is to say, we need to know how many credits have defaulted before time t. The following lemma provides the formula needed to determine this.

Lemma 4.1. The NPV for CDO conditioning on the information that m credits defaulted before time t is:

$$NPV(t,m) = S \times Premium_{A,B}(t,T_0,T_b) - Default_{A,B}(t,T_0,T_b),$$

where the $Default(t,T_0,T_b)$ and $Premium(t,T_0,T_b)$ are given by (20) and,

$$E_t\left[L_{A,B}[n(u)]\right] = \sum_{n(u)=m}^{M} L_{A,B}[n(u)]\pi(n(u),M,u|m,t),$$

$$\pi(n(u),M,u|m,t) = E\left[\frac{(M-m)!}{(n(u)-m)!(M-n(u))!}q(u|t)^{n(u)-m}[1-q(u|t)]^{M-n(u)},\right]$$

in that:

$$q(u|t) = \Phi\left(\frac{\Phi^{-1}[1-\frac{\mathbb{Q}(\tau_i>u)}{\mathbb{Q}(\tau_i>t)}] - aF - bC}{\sqrt{1-a^2-b^2-2\rho ab}}\right).$$

The proof for this is as follows: From the protection seller's point of view, the NPV of the CDO contract is equal to the expectation of discounted cash flows received minus cash flows paid; this yields the first equation. Furthermore, in equation (20), the only unknown quantity is the expected tranche loss $E_t[L_{A,B}[n(u)]]$; however, it could be solved by the following equation:

$$E_t\left[L_{A,B}[n(u)]\right] = \sum_{n(u)=m}^{M} L_{A,B}[n(u)]\mathbb{Q}(n(u)\ default\ by\ u|m\ default\ by\ t).$$

The risk-neutral probability of n defaults by u, given that m credits default before time t, should be handled with caution. Knowing the information that m credits default before time t is equivalent to knowing that there are $M-m$ credits remaining in the portfolio and all of them are alive at t. Therefore, we have to compute the conditional default probability for each name. This can be calculated as follows:

$$\begin{aligned}\mathbb{Q}(\tau_i < u|\tau_i > t) &= \frac{Q(t<\tau_i<u)}{\mathbb{Q}(\tau_i>t)}\\ &= \frac{\mathbb{Q}(\tau_i>t)-\mathbb{Q}(\tau_i>u)}{\mathbb{Q}(\tau_i>t)}\\ &= 1-\frac{\mathbb{Q}(\tau_i>u)}{\mathbb{Q}(\tau_i>t)}.\end{aligned}$$

The survival probability $\mathbb{Q}(\tau_i <,)$ can be computed using (8). This is referred to as the conditional default probability for each surviving credit, and we can construct the default dependence structure between these surviving credits by using the two-factor Gaussian copula model discussed in section 3.2. This leads to a default rate distribution conditional on F and C as follows:

$$\pi(n(u), M, u|m, t, F, C) = \frac{(M-m)!}{(n(u)-m)!(M-n(u))!} q(u|t)^{n(u)-m}[1-q(u|t)]^{M-n(u)},$$

where

$$q(u|t) = \Phi\left(\frac{\Phi^{-1}[1-\frac{\mathbb{Q}(\tau_i>u)}{\mathbb{Q}(\tau_i>t)}] - aF - bC}{\sqrt{1-a^2-b^2-2\rho ab}}\right),$$

By taking the expectation of $\pi(n(u), M, u|m, t, F, C)$, we can complete the proof for Lemma 4.1. □

Once we have obtained the formula for NPV, we are ready to complete the calculation for CVA. If we go back to the CVA formula (equation (3)), we are left with the need to determine the expectation for the integral. To solve this, we can apply the two-factor Gaussian copula model once again in order to compute the probability that m will default prior to time t conditional on the information available currently and on the default time of the counterparty. The following theorem formulates the results:

Theorem 4.2. The CVA for the CDO contract considered above from the protection seller's point of view is given by:

$$CVA(0, T_b) = Loss_C \int_0^{T_b} E_0\left(D(0,t)[NPV(t)]^+|\tau_C = t\right) d_t\mathbb{Q}(\tau_C \le t),$$

and the expectation for the integral is represented as:

$$E_0\left(D(0,t)[NPV(t)]^+|\tau_C = t\right) = \sum_{m=0}^{M} P(0,t)NPV(t,m)^+\pi(m,t|C = \Phi^{-1}[H_C(t)]),$$

where:

$$\pi(m,t|C = \Phi^{-1}[H_C(t)]) = \int_{-\infty}^{\infty} q(m,t|f, \Phi^{-1}[H_C(t)])g(f|\Phi^{-1}[H_C(t)])df,$$

and:

$$q(m,t|f,c) = \Phi\left(\frac{\Phi^{-1}[1-\mathbb{Q}(\tau_i < t)] - a\times f - b\times c}{\sqrt{1-a^2-b^2-2\rho ab}}\right),$$

$$g(f|c) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} exp\left[-\frac{(f-\rho c)^2}{2(1-\rho^2)}\right].$$

This is proven as follows: We begin with the expectation. Since the NPV depends entirely on the default rate distribution conditioning on the counterparty's default time information, we can express the expectation as follows:

$$\sum_{m=0}^{M} NPV(t,m)^{+} \times \mathbb{Q}(m \text{ default before t}|\tau_C = t).$$

It can be difficult to handle the information that the counterparty defaults exactly at time t. However, thanks to the two-factor Gaussian copula model proposed above, it is easy to convey this information by means of the default rate distribution (18):

$$\{\tau_C = t\} \equiv C = \Phi^{-1}[H_C(t)] = \Phi^{-1}[1 - \mathbb{Q}(\tau_C \leq t)].$$

Consequently, the probability given the default information is transformed into the probability given the value of C. This can be solved by evaluating the conditional expectation of equation (15). Note that given that (F, C) is bivariate normally distributed with zero mean, unit variance and correlation coefficient ρ, the conditional density of F given C is again a normal distribution with mean $\rho \times C$ and variance $1 - \rho^2$. As a result, the conditional expectation can be evaluated by:

$$\pi(m,t|C = \Phi^{-1}[H_C(t)]) = \int_{-\infty}^{\infty} q(m,t|f, \Phi^{-1}[H_C(t)])g(f|\Phi^{-1}[H_C(t)])df.$$

$\square$

We have now completed our derivation of CVA for synthetic CDO tranches, and the integral in the formula can be approximated by a sufficiently small time partition in the Riemann sense. We defer the default of the counterparty to the first bucket time following it if it occurs between the bucket dates. Note that the above framework is not appropriate for the case that payment dates are included in the bucket dates. A more general case is discussed in the appendix.

5 Model Calibration and Implementation

Before we begin our numerical analysis, it should prove useful to briefly discuss our methods of model calibration and implementation. In this section, we first address how our model should be calibrated. Following that, we suggest a model-implementation procedure that can be used in order to compute CVAs for CDOs.

5.1 Model Calibration

Let us first consider the Shifted Squared Root Diffusion (SSRD) stochastic intensity model introduced in the previous section. In order to calibrate our model, it is necessary that we know the survival probabilities implied in the market. In general, these probabilities can be determined by considering the quotes of CDSs referencing the counterparty or by referring to the corporate bond market. In our case, we chose the former method.

Consider a CDS written on the counterparty that starts at T_0 and has a running spread (R_{CDS}) that matures at T_r. If we assume that the interest rate is deterministic, the price of a CDS can be expressed as:

$$\begin{aligned} CDS(0,T_0,T_r,REC_C) = R_{CDS}\Bigg(&-\int_{T_0}^{T_r} P(0,u)(u-T_{\omega(u)-1})d_u\mathbb{Q}(\tau_C \geq u) \\ &+\sum_{k=1}^{r} P(0,T_k)\delta_i\mathbb{Q}(\tau_C \geq T_k)\Bigg) \\ &+(1-REC_C)\int_{T_0}^{T_r} P(0,u)d_u\mathbb{Q}(\tau_C \geq u), \end{aligned} \tag{22}$$

where $T_{\omega(u)-1}$ is the closest T_k prior to u, and δ_k is the time frame between time T_{k-1} and T_k. This formula is model independent, which allows us to extract survival probabilities from market CDS quotes without following any particular model specification. To calibrate the intensity model, we must specify the following:

- the functional form of $\int_t^T \beta(u;\kappa,\mu,\nu,\lambda_0)du$, thereby ensuring that the intensity model exactly fits the survival probabilities we extract from market CDS quotes, and
- the value of $(\kappa,\mu,\nu,\lambda_0)$ such that the implied volatility surface reproduced by the model is as close as possible to the market implied volatility surface of single name credit derivatives.

Let $\mathbb{Q}_{mkt}(\tau_C > t)$ denote the survival probability that has been extracted from CDS quotes. We select $\int_0^t \beta(u;\kappa,\mu,\nu,\lambda_0)du$ such that $\mathbb{Q}_{mkt}(\tau_C > t) = \mathbb{Q}(\tau_C > t|\mathcal{G}_0)$. This implies the following specific formulation:

$$\int_0^t \beta(u;\kappa,\mu,\nu,\lambda_0)du = \ln\frac{Survival(0,t;\kappa,\mu,\nu,\lambda_0)}{\mathbb{Q}_{mkt}(\tau_C > t)}. \tag{23}$$

In this way, our intensity model is exactly calibrated to the market implied survival probabilities. We can then make use of the remaining parameters to calibrate the single name default option data. However, this may prove to be a dangerous move if the options market is not liquid enough. In that case, we should either refer to complex index derivatives or consider those quotes with caution. In order to avoid parameter instability, we have chosen not to refer to further option data here. To specify our parameters, we set some possible values of $(\kappa, \mu, \nu, \lambda_0)$ such that the model can produce reasonable option-implied volatilities.

Calibration of the two-factor Gaussian copula model may prove problematic owing to the difficulty of calibrating ρ. The only possible proxies for the correlation parameter ρ would be the contingent CDS (CCDS) or related products, and calibration would be easier if the CCDS or related instruments were liquid. However, this is often not the case. To calibrate our model without these contingent products, we have attempted to ensure that our model matches the liquid prices of CDX NA IG index swaps and standardized CDX NA IG index tranches as much as possible. Therefore, we have calibrated our model according to the following steps:

- Set $C = c_0$, where c_0 is large enough such that $\Phi(c_0)$ is close to 1 (we set $c_0 = 3$). This means that we regard the quoted prices as default-free prices.[3]
- Set $\rho = 0$. This is reasonable when the counterparty is assumed to have no default risk. The infinite default time is not dependent on any risk factors.
- Adjust α, θ, σ, h_0, a, and b so that the model prices fit the market quotes of tranches and other liquid index swaps as much as possible.
- Select a for each tranche so that the model price is the same as the market price. Note that we have not referred to a base correlation in our two-factor model, since it is often not feasible to find a base correlation for each detachment point.

[3] The bid-offer spreads of CDO quotes from dealers should be able to cover funding cost, liquidity premium, and CCR. At this stage, we have decreased the bid-offer spread to 0 in order to place ourselves in a milieu that does not take funding cost or liquidity risk into consideration and that assumes a default-free counterparty.

Once the model has been calibrated to the market instruments, we can select a suitable ρ for the counterparty. This is not easy to do; it is often selected based on the acquired knowledge of experienced risk managers.

5.2 Model Implementation

In this subsection, we would like to give some recommendations to risk managers who carry out the valuation of the CCR of synthetic CDO tranches. When a risk manager is responsible for calculating the CVA of a tranche, we suggest the following procedure:

- Calibrate the model following the steps suggested in the previous subsection.
- Select an appropriate value for the factor correlation, ρ. This often proves to be difficult, since there is no proxy for this parameter. The manager's personal experience and professionalism may help. [4]
- Apply the formulas discussed in the previous section and in the appendix in order to compute the CVA.

We can also apply this model to evaluate CVA for tranches with non-standard attachment or detachment points and non-standard maturity. Furthermore, the model can be applied to a netted portfolio of CDO tranches that references the same credit portfolio.

6 Numerical Examples

In this section, we would like to consider some numerical examples in order to test the impact of the correlation coefficient ρ (factor correlation) on synthetic CDO tranches across different attachment and detachment points. First of all,

[4] One may view the factor correlation as the correlation between the macroeconomic state and the counterparty's credit quality. However, it should not be estimated historically by calculating the relationship between some economic variable and the variable mirroring the counterparty's credit quality. This is because the model works in terms of a risk-neutral probability measure rather than in terms of a physical or real world probability measure.

please note that an increase in factor correlation results in an increase in default correlation between underlying credit and the counterparty. In order to calculate wrong-way risk, we used two hypothetical default probability term structures for the counterparty. This is shown in Table 1. The intensity model is calibrated to these survival term structures.

On the other hand, we calibrated our two-factor Gaussian copula model using the CDX NA IG series 15 tranche quotes from March 9, 2011, using the procedure suggested in the previous section. Due to the difficulty of fitting super senior tranches, for our calibration inputs, we chose to use only prices of 0% - 3%, 3% - 7%, and 7% - 15% with maturities 3, 5, and 7 years.

Let us consider the case of a default-free investor who acts as a protection buyer, while her counterparty acts as a protection seller. In this case, the investor exposes herself to the risk that the counterparty will fail to provide protection on the portfolio loss. Table 2 and Figure 1 report the numerical results of CVA associated with different values of factor correlations across different attachment and detachment points. As the table makes clear, the factor correlation has a significant impact on the CVA, and ignoring the correlation may lead to a misestimation of CCR. In particular, factor correlation has a negative impact on CVA. This may be because a higher factor correlation, or a higher default correlation, is associated with a higher probability that all credits will default at the same time. In other words, an increase in factor correlation involves a corresponding decrease in the probability that a small number of credits will default at once. Table 3 shows the impact of factor correlation on the default rate density when $n \leq 7$. Because equity tranche is responsible for the losses of the first three entities, this correlation is critical for equity tranche valuation and for the corresponding CVA. The lower the default probabilities are, the lower is the expected residual NPV from the viewpoint of the protection buyer. Consequently, the default option for the counterparty becomes more out-of-the-money, and the option value (i.e., the CVA) decreases as correlation increases.

Note further that when the counterparty's quality deteriorates, i.e., when the probability of default increases significantly, (as when the term structure of default probability shifts from "Low Risk" to "High Risk" in Table 1), the CVA associated with higher factor correlations increases less than the CVA associated with lower or negative factor correlations. This means that in the case of a higher default correlation between the counterparty and their underlying credits, (the counterparty and the underlying credits are more likely to default together), counterparty exposure decreases and thus renders CCR less relevant.

We then consider the case of a default-free investor who acts as a protection seller, while her counterparty acts as a protection buyer. In this situation, the the protection seller is likely to worry that the counterparty will fail to pay their protection fee. Table 4 and Figure 2 report the results of CVA. We found that these CVAs are generally smaller than those of the protection buyers except with regard to the senior tranche (7%-15%). This may indicate that the upfront payment acts as a deterrent, preventing the protection seller from suffering the results of the default of the protection buyer. However, for senior tranches, since the upfront fee is often small or negative, this effect does not exist. Factor correlation is not as important in this case as it is in the case of the protection buyer with the exception of the case of the equity tranche, which has a positive influence on CVA. This positive effect is due to the fact that default rate probabilities decrease when correlations increase. The lower the default rate probabilities are, the higher is the expected residual NPV. The default option of the counterparty becomes more in-the-money, and thus, the CVA increases. Note that where there is a very high level of factor correlation ($\rho = 0.99$), the CVA for the equity tranche increases more significantly, as in the case of counterparty deterioration. This suggests that selling protection on the most risky equity tranches exposes the seller to the default risk of underlying credits as well as to wrong-way risk. For this reason, it is necessary to proceed with caution when dealing with such tranches.

7 Conclusions

In this study, we have proposed a way to evaluate CCR for synthetic CDO tranches. Our approach is quite innovative in that we consider CCR with respect to the interrelationship between the portfolio and the counterparty. We have characterized these effects based on the two-factor Gaussian copula model and the

SSRD stochastic intensity model. Our framework provides an analytical approximation for CVAs for CDO tranches. This has the benefit of reducing computation times. At the same time, the modeling framework we suggested is also beneficial in that it is always a viable choice for analyzing counterparty risk with regard to other multi-name credit derivatives. Finally, we have presented some numerical tests that are designed to examine the impact of default correlation on the CVA of the CDX NA IG index tranches.

We found that the results of our numerical tests differed based on whether we considered the data from the point of view of protection buyers or protection sellers. For protection buyers, we discovered that the default correlation greatly impacts the valuation of CCR. In particular, we found that this relationship has a negative influence on CVA. This suggests that CCR is less relevant when there is a high degree of positive correlation between the counterparty and the underlying credit. On the other hand, for protection sellers, this correlation does not appear to have such a significant impact, except in the case of equity tranches. With respect to equity tranches, an extremely positive correlation has a positive impact on the CVA, and the CVA increases even more significantly when the counterparty deteriorates. This suggests that one should deal with CCR with caution if one is selling protection for the most risky type of tranche.

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Appendix

We would like to consider another general case for the computation of CVA for CDO tranches. Suppose that we have the following bucket dates $[t_1, ...t_n]$, and assume that some of these are the payment dates, i.e., $[t_1, ...t_n] \cap [T_1, ..., T_b] \neq \phi$. In the case of $t_i = T_j$, for some i and j, the expectation for the integral cannot be computed as formulated in section 4.2, since the NPV contains discounted future cash flows as well as cash flows received today. In order to compute the cash flow received today, we need to know the default rate information for the previous payment date. For this reason, we should compute the NPV of CDO tranches whenever k defaults by $T_{w(t_i)-1}$ and m defaults by $T_{w(t_i)}$ for all $k \leq m$. In formal terms, this expectation can be evaluated by:

$$\sum_{m=0}^{M} \sum_{k=0}^{m} D(0,t) NPV(t,m,k)^{+} \mathbb{Q}(k \text{ by } T_{w(t_i)-1}, m \text{ by } T_{w(t_i)} | \tau_C = T_{w(t_i)}), \qquad (24)$$

and the probability $\mathbb{Q}(k \text{ by } v, m \text{ by } u | \tau_C = u)$ can be expressed as:

$$\mathbb{Q}(m \text{ by } u | k \text{ by } v, \tau_C = u) \mathbb{Q}(k \text{ by } v | \tau_C = u).$$

Thanks to the two-factor Gaussian copula model, this probability can be evaluated easily and quickly (see section 4.2). On the other hand, the NPV will be divided into two parts; these are the exact amount received today and and cash flows anticipated to be received in the future. Given k defaults before $T_{w(t_i)-1}$ and m defaults before $T_{w(t_i)}$, the amount received today is:

$$S \times Premium - Default,$$

and:

$$Default = L_{A,B}[n(T_{w(t_i)})] - L_{A,B}[n(T_{w(t_i)-1})],$$
$$Premium = \delta_w(t_i) \left[1 - \frac{L_{A,B}[n(T_{w(t_i)-1})] + L_{A,B}[n(T_{w(t_i)})]}{2} \right].$$

The cash flows anticipated to be received in the future could be calculated similarly to the NPV introduced in section 4.2, except that we would have to start counting cashflow at $T_{w(t_i)+1}$.

Payment dates should be included in the bucket dates. It is often the case that the counterparty defaults right before a payment date; therefore, the procedure introduced here should be applied.

Table 1: Default Probability Term Structure

Maturity	Low Risk (%)	High Risk (%)
6 months	0.47	0.47
1 year	0.91	0.91
2 years	3.38	8.00
3 years	6.75	12.00
4 years	11.83	22.00
5 years	17.98	31.00
7 years	27.55	45.00
10 years	39.52	57.00

This table presents the default probabilities term structure associated with counterparties that have low (first column) and high (second column) default risk.

Table 2: Protection Buyer's CVA for 3-year CDX NA IG Tranches

ρ	$0\% - 3\%$	$3\% - 7\%$	$7\% - 15\%$
ind CVA	39.1688(79.431)	27.2634(55.9197)	1.20605(2.4981)
-0.99	74.2662(152.898)	39.1291(80.2551)	2.40226(4.66442)
-0.5	55.6249(114.409)	33.7388(69.032)	2.12456(4.18265)
0.0	37.8679(76.7223)	28.6001(58.3073)	1.86851(3.7284)
0.5	22.4033(42.2808)	23.8625(48.3781)	1.62904(3.3004)
0.99	10.3074(16.0528)	19.6463(39.4787)	1.4161(2.91209)

The table presents credit value adjustmenst for 3-year CDX NA IG tranches given the assumption that the investor is the protection buyer. Each row reports the CVA associated with the corresponding ρ of each tranche for both low and high (in round brackets) counterparty default risk. The "ind CVA," (first row) reports CVA without considering dependence as shown in equation (4).

Table 3: Default Rate Probabilities

		Default Rate Probability (%)				
Maturity	n	$\rho = -0.99$	$\rho = -0.5$	$\rho = 0.0$	$\rho = 0.5$	$\rho = 0.99$
T_1	1	0.052	0.041	0.028	0.014	0.005
	2	0.005	0.006	0.005	0.003	0.003
	3	0.001	0.002	0.002	0.002	0.002
	4	0.000	0.001	0.001	0.001	0.001
	5	0.000	0.000	0.001	0.001	0.001
	6	0.000	0.000	0.000	0.001	0.000
T_2	1	1.721	1.195	0.764	0.421	0.185
	2	0.413	0.321	0.227	0.139	0.080
	3	0.178	0.150	0.114	0.080	0.053
	4	0.102	0.091	0.074	0.058	0.039
	5	0.067	0.063	0.055	0.045	0.029
	6	0.046	0.047	0.042	0.034	0.022
T_3	1	4.060	2.839	1.849	1.087	0.506
	2	1.263	0.936	0.652	0.404	0.202
	3	0.597	0.473	0.347	0.224	0.117
	4	0.336	0.285	0.219	0.149	0.078
	5	0.211	0.189	0.153	0.108	0.057
	6	0.143	0.133	0.112	0.081	0.044

The above table shows default rate probabilities computed from the two-factor Gaussian copula model along with the corresponding number of defaults (n), at maturity T_1 (21 May, 2011), T_2 (20 Jun, 2011), and T_3 (20 Sep, 2011), and different factor correlations ρ.

Table 4: Protection Seller's CVA for 3-year CDX NA IG Tranches

ρ	$0\% - 3\%$	$3\% - 7\%$	$7\% - 15\%$
ind CVA	9.85274(11.6853)	2.74797(3.25961)	8.86679(15.6942)
-0.99	4.80741(5.82627)	2.09725(2.53459)	8.14524(14.3843)
-0.5	6.43261(7.79901)	2.31539(2.79879)	8.31242(14.7746)
0.0	9.00959(10.9262)	2.54913(3.08137)	8.47738(15.1452)
0.5	13.0899(15.9328)	2.81326(3.40115)	8.62691(15.4794)
0.99	18.8355(25.8811)	3.12521(3.77889)	8.76546(15.7799)

The table presents credit value adjustments for 3-year CDX NA IG tranches given the assumption that the investor is the protection seller. Each row reports the CVA associated with the corresponding ρ of each tranche for both low and high (in round brackets) counterparty default risk. The "ind CVA," (first row) reports CVA without considering dependence as shown in equation (4).

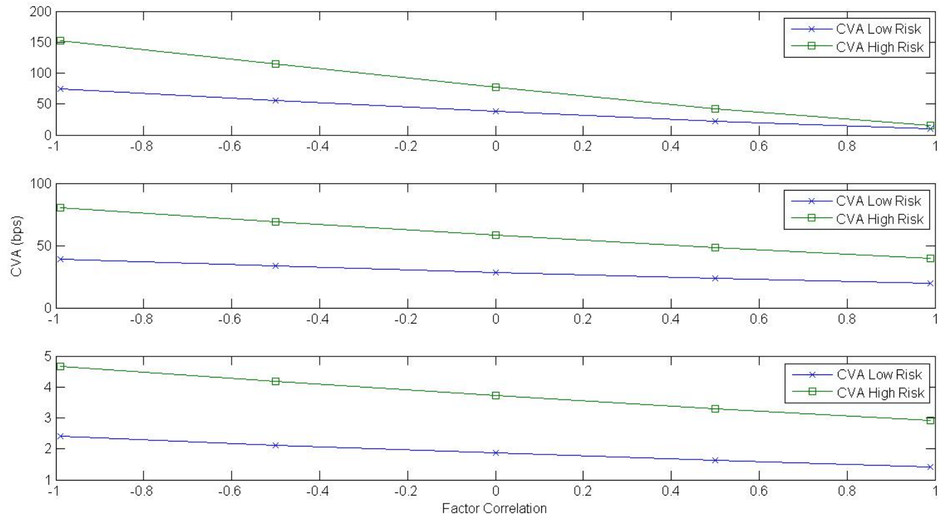

Figure 1: Protection Buyer's CVA of CDX NA IG Tranches

Here we see CVAs of 0% - 3% (top graph), 3% - 7% (middle graph), and 7% - 15% (bottom graph) tranches associated with different values of factor correlation as seen from the protection buyer's view..

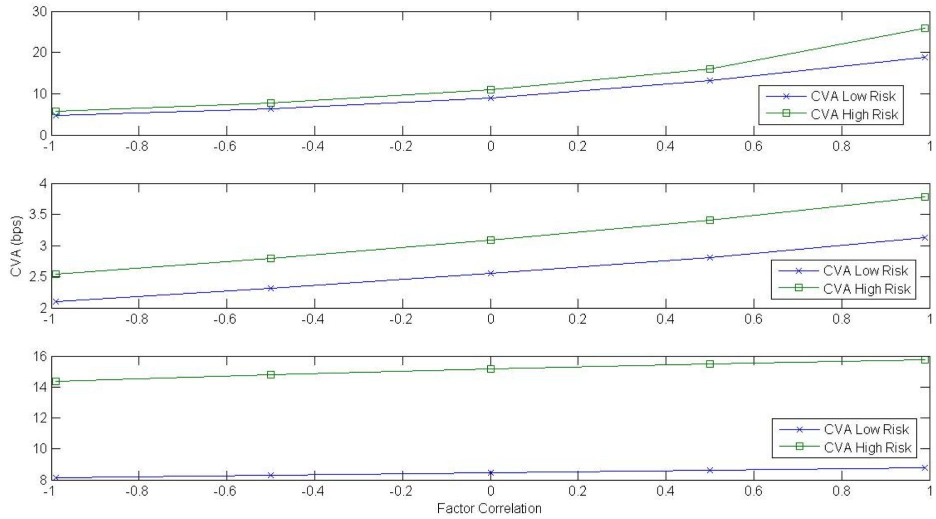

Figure 2: Protection Seller's CVA of CDX NA IG Tranches

Here we see CVAs of 0% - 3% (top graph), 3% - 7% (middle graph), and 7% - 15% (bottom graph) tranches associated with different values of factor correlation as seen from the protection seller's view..